

## 9 More on the 1D Heat Equation

### 9.1 Heat equation on the line with sources: Duhamel's principle

**Theorem:** Consider the Cauchy problem

$$\begin{aligned}\frac{\partial u}{\partial t} &= D \frac{\partial^2 u}{\partial x^2} + F(x, t), \quad \text{on } |x| < \infty, t > 0 \\ u(x, 0) &= f(x) \text{ for } |x| < \infty\end{aligned}\tag{1}$$

where  $f$  and  $F$  are defined and integrable on their domains. Let  $S(x, t) = \frac{1}{2\sqrt{\pi Dt}} e^{-x^2/4Dt}$  be the usual fundamental solution to the heat equation. Then the solution to (1) is given by

$$\begin{aligned}u(x, t) &= \int_{-\infty}^{\infty} S(x - y, t) f(y) dy + \int_0^t \int_{-\infty}^{\infty} S(x - y, t - \tau) F(y, \tau) dy d\tau \\ &= \frac{1}{2\sqrt{\pi Dt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4Dt} f(y) dy \\ &\quad + \int_0^t \frac{1}{2\sqrt{\pi D(t-\tau)}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4D(t-\tau)} F(y, \tau) dy d\tau.\end{aligned}\tag{2}$$

For motivation consider the ODE case of a “forced” IVP:

$$\begin{aligned}\frac{dy}{dt} + ay &= F(t), \quad \text{for } t > 0, a \text{ is a constant} \\ y(0) &= 0\end{aligned}\tag{3}$$

It is straightforward to apply the integrating-factor method, or Laplace transform method to obtain

$$y(t) = \int_0^t e^{-a(t-\tau)} F(\tau) d\tau.\tag{4}$$

Now consider a one-parameter set of homogeneous equations

$$\begin{aligned}\frac{dw}{dt} + aw &= 0, \quad \text{for } t > 0 \\ w(0; \tau) &= F(\tau)\end{aligned}\tag{5}$$

Here  $\tau$  is just a dummy parameter with respect to this problem since the derivative is respect to  $t$ ; the solution to this problem, for each  $\tau$ , is  $w(t) = w(t; \tau) = F(\tau)e^{-at}$ . By observation the solution (4) to the source problem (3) is the integral of the solution  $w$  to the homogeneous problem (5) with  $t$  replaced by  $t - \tau$ :

$$y(t) = \int_0^t w(t - \tau; \tau) d\tau.$$

Put another way, the solution to the nonhomogeneous equation, with homogeneous initial condition, is deduced from the solution to the homogeneous equation (with appropriately parameterized nonhomogeneous initial condition). This is *Duhamel's principle*, and it is fairly generalizable. So, can this idea carry over to diffusion equations? From previous work solving the heat equation problem we know that the problem

$$w_t = Dw_{xx} \quad \text{on } |x| < \infty, t > 0$$

$$w(x, 0; \tau) = F(x, \tau) \text{ on } |x| < \infty$$

has the solution  $w(x, t; \tau) = \int_{-\infty}^{\infty} S(x - y, t) F(y, \tau) dy$  (again,  $\tau$  is just a parameter).

Now consider the problem

$$u_t = Du_{xx} + F(x, t) \quad \text{on } |x| < \infty, t > 0 \tag{6}$$

$$u(x, 0) = 0 \text{ in } |x| < \infty.$$

By applying the above logic from the ODE case, the solution  $u(x, t)$  should be the time integral of  $w$  with  $t$  replaced by  $t - \tau$ :

$$u(x, t) = \int_0^t w(x, t - \tau; \tau) d\tau = \int_0^t \int_{-\infty}^{\infty} S(x - y, t - \tau) F(y, \tau) dy d\tau \tag{7}$$

By taking derivatives of (7) it is straight forward to verify that (7) is a solution to (6). Since (2) represents the solution to (6) plus the solution to the IVP with homogeneous equation, non-zero initial condition, then, by linearity, (2) is the sought after solution to (1).

*Example 1:* Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + xt \quad \text{on } |x| < \infty, t > 0$$

$$u(x, 0) = 0 \text{ for } |x| < \infty$$

By the theorem,  $u(x, t) = \int_0^t \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4(t-\tau)}}{2\sqrt{\pi(t-\tau)}} y \tau dy d\tau$ .

Note that, with  $r = (x - y)/2\sqrt{t - \tau}$ ,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4(t-\tau)}}{2\sqrt{\pi(t-\tau)}} y dy &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x - 2r\sqrt{t-\tau}) e^{-r^2} dr \\ &= \frac{x}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-r^2} dr - \frac{2\sqrt{t-\tau}}{\sqrt{\pi}} \int_{-\infty}^{\infty} r e^{-r^2} dr \\ &= x \end{aligned}$$

since the first integral in the last expression is  $\sqrt{\pi}$ , and the last integral is 0 (the integrand is an odd function on a symmetric interval...or just integrate it directly by substitution). Hence,  $u(x, t) = \int_0^t x \tau d\tau = xt^2/2$ .

*Example 2:* Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2x \sin(t) \quad \text{on } |x| < \infty, t > 0$$

$$u(x, 0) = e^{-a|x|} \text{ for } |x| < \infty$$

Write  $u(x, t) = u^{(1)}(x, t) + u^{(2)}(x, t)$ , where  $u^{(i)}, i = 1, 2$  solve the problems

$$\frac{\partial u^{(1)}}{\partial t} = \frac{\partial^2 u^{(1)}}{\partial x^2} \quad \text{on } |x| < \infty, t > 0$$

$$u^{(1)}(x, 0) = e^{-a|x|} \text{ for } |x| < \infty$$

and

$$\frac{\partial u^{(2)}}{\partial t} = \frac{\partial^2 u^{(2)}}{\partial x^2} + 2x \sin(t) \quad \text{on } |x| < \infty, t > 0$$

$$u^{(2)}(x, 0) = 0 \text{ for } |x| < \infty.$$

By using the solution representation to the regular (homogeneous) heat equation Cauchy problem, the reader should verify that

$$u^{(1)}(x, t) = \frac{e^{a^2 t}}{\sqrt{\pi}} \left\{ e^{-ax} \int_{a\sqrt{t}-x/2\sqrt{t}}^{\infty} e^{-r^2} dr + e^{ax} \int_{-\infty}^{-a\sqrt{t}-x/2\sqrt{t}} e^{-r^2} dr \right\}.$$

For the  $u^{(2)}$  problem, consider the problem

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} \text{ on } |x| < \infty, t > 0$$

$$w(x, 0) = 2x \sin(\tau) \text{ for } |x| < \infty.$$

Again, from the solution to the Cauchy problem,  $w(x, t) = w(x, t; \tau) = 2x \sin(\tau)$ . By Duhamel's principle,

$$u^{(2)}(x, t) = \int_0^t 2x \sin(t - \tau) d\tau = 2x \int_0^t \sin(t') dt' = 2x(1 - \cos(t)).$$

**Summary:** The point of this section is knowing solution form (2) to the IVP (1).

### Exercises

1. Consider

$$u_t = Du_{xx} + 3e^{-x^2} \text{ on } |x| < \infty, t > 0$$

$$u(x, 0) = \text{sech}(2x) \text{ for } |x| < \infty$$

It is rare that the integrals associated with the solution of Cauchy heat equation problems can be calculated out analytically, so in this case write out the solution from the theorem without carrying out the specific integrations.

2. Let  $F(x, \tau)$  be an arbitrary bounded, continuous function on  $\mathbb{R}^2$ . Define

$$v(x, t, \tau) := \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4D(t-\tau)}}{2\sqrt{\pi D(t-\tau)}} F(y, \tau) dy \quad \text{for } t > \tau$$

Verify that  $u(x, t) = \int_0^t v(x, t, \tau) d\tau$  satisfies  $u_t = Du_{xx} + F(x, t)$ .

3. Consider the Cauchy problem

$$u_t = u_{xx} - a \quad \text{on } |x| < \infty, t > 0, \text{ with } a \text{ being a constant}$$

$$u(x, 0) = H(x) \quad \text{for } |x| < \infty, \text{ where } H(\cdot) \text{ is the Heaviside function}$$

First, solve this using linearity and Duhamel's principle. Second, solve directly by finding a transformation that reduces the problem to a homogeneous heat equation problem.

4. Consider  $u_t = u_{xx} + F(x, t)$ ,  $|x| < \infty$ ,  $t > 0$ , where now  $F(x, t) \geq 0$  everywhere. Assume  $u(x, 0) = 0$ ,  $|x| < \infty$ , and  $u \rightarrow 0$  as  $|x| \rightarrow \infty$ . Show that  $u(x, t) \geq 0$  everywhere on its domain.

## 9.2 Higher dimensional Cauchy problems

For the case of  $\mathbf{x} \in \mathbb{R}^n$ , consider the problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= D \nabla^2 u = D \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} \quad \text{for } \mathbf{x} \in \mathbb{R}^n, t > 0 \\ u(\mathbf{x}, 0) &= f(\mathbf{x}) \quad \text{for } \mathbf{x} \in \mathbb{R}^n \end{aligned} \quad (8)$$

The fundamental solution in 1D,  $S(\mathbf{x}, t)$ , has a direct analogue in higher dimensions, namely

$$S(\mathbf{x}, t) = S(x_1, x_2, \dots, x_n, t) = \frac{1}{(4\pi Dt)^{n/2}} e^{-|\mathbf{x}|^2/4Dt}, \text{ where } |\mathbf{x}| = \sqrt{x_1^2 + x_2^2 \dots x_n^2}.$$

So the solution to (8) is

$$u(\mathbf{x}, t) = \int_{\mathbb{R}^n} S(\mathbf{x} - \mathbf{y}, t) f(\mathbf{y}) d\mathbf{y} = \frac{1}{(4\pi Dt)^{n/2}} \int_{\mathbb{R}^n} e^{-|\mathbf{x} - \mathbf{y}|^2/4Dt} f(\mathbf{y}) dy_1 dy_2 \dots dy_n.$$

Again considering a Cauchy problem with sources

$$\frac{\partial u}{\partial t} = D \nabla^2 u + F(\mathbf{x}, t), \quad \mathbf{x} \in \mathbb{R}^n, t > 0$$

$$u(\mathbf{x}, 0) = f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n.$$

We now have the analogous solution, again by Duhamel's principle,

$$u(\mathbf{x}, t) = \int_{\mathbb{R}^n} S(\mathbf{x} - \mathbf{y}, t) f(\mathbf{y}) d\mathbf{y} + \int_0^t \int_{\mathbb{R}^n} S(\mathbf{x} - \mathbf{y}, t - \tau) F(\mathbf{y}, \tau) d\mathbf{y} d\tau.$$

## 9.3 Duhamel's principle and the wave equation

Recall the theorem in Section 6.2 that for the forced wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx} + F(x, t) & |x| < \infty, t > 0 \\ u(x, 0) = f(x), u_t(x, 0) = g(x) & |x| < \infty, \end{cases} \quad (9)$$

the explicit solution is given by

$$u(x, t) = \frac{1}{2} \{f(x+ct) + f(x-ct)\} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y) dy + \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(y, \tau) dy d\tau . \quad (10)$$

This was proved in the appendix 6.3.2 via Green's theorem. We can obtain the same result by employing Duhamel's principle. Consider

$$\begin{cases} v_{tt} = c^2 v_{xx} + F(x, t) & |x| < \infty, t > 0 \\ v(x, 0) = 0, v_t(x, 0) = 0 & |x| < \infty \end{cases} \quad (11)$$

Obtaining the solution to this problem just means adding on d'Alembert's solution to obtain the solution to (9) (namely (10)). Thus, consider the problem

$$\begin{cases} w_{tt} = c^2 w_{xx} & |x| < \infty, t > 0 \\ w(x, 0) = 0, w_t(x, 0; \tau) = F(x, \tau) & |x| < \infty \end{cases} \quad (12)$$

From d'Alembert's solution to this homogeneous equation problem,

$$w(x, t; \tau) = \frac{1}{2c} \int_{x-ct}^{x+ct} F(y, \tau) dy . \quad (13)$$

Then, following the same logic as in subsection 9.1,

$$v(x, t) = \int_0^t w(x, t - \tau; \tau) d\tau = \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(y, \tau) dy d\tau , \quad (14)$$

which agrees with the former theorem.

*Exercise:* Duhamel's principle works just as well with first-order hyperbolic equations. Consider the problem

$$u_t - u_x = e^{-x-t} \quad |x| < \infty, t > 0$$

$$u(x, 0) = 0 \quad |x| < \infty$$

1. Use Duhamel's principle<sup>1</sup> to obtain the solution to the non-homogeneous problem.

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<sup>1</sup>Hence, you need to solve the problem  $w_t - w_x = 0$ ,  $w(x, 0) = w(x, 0; \tau) = e^{-x-\tau}$ .

2. Solve the problem via the characteristic equation method discussed in Section 4 of these Notes. The two approaches should give you the same answer.  
(Ans:  $u(x, t) = te^{-(x+t)}$ .)